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Binary decompositions and varieties of sums of binaries

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Abstract

A homogeneous element in $\mathbb{C}[x_0, \ldots, x_n]$ is a binary form if it "essentially" involves two variables. The problem of decomposing a generic form as a sum of binary forms is introduced and studied. In the $n = 2$ case a complete answer to a Waring type problem is given and the space parameterizing all the binary decompositions of a generic form is studied: its dimension and its degree are determined and a more detailed description is given for forms of degree $d = 2, 3, 5, 8$.

MSC: 14N15; 51N35

1. Introduction

Let $S = \mathbb{C}[x_0, \ldots, x_n]$ and consider $f \in S_d$ a homogeneous element of degree d. Usually f is presented as a sum of monomials, but other additive decompositions are possible. For a fixed natural number $m \leqslant n$, one can consider decompositions of f of the following kind:

$$f = f_1 + \cdots + f_s$$

such that there exist linear forms $y_{1,i}, \ldots, y_{m,i} \in S_1$ for which f_i is a degree d form in $\mathbb{C}[y_{1,i}, \ldots, y_{m,i}] \subset S$, for $i = 1, \ldots, s$. In other words, this is a decomposition of f as sum of

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forms “essentially” involving a smaller number of variables. For $m = 1$ this is just the usual sum of powers decomposition for which there is a vast literature (see below). For $m = 2$ this is called a *binary decomposition* of f and the f_i's are called *binary forms*. To stress the fact that f_i is a binary form, we will sometimes use the notation $f_i(\cdot, \cdot)$. Notice, for example, that $x_0^d + (x_1 + \cdots + x_n)^d$ is a binary form, while $x_0^d + \cdots + x_n^d$ is not as soon as $n \geqslant 2$. In this paper, we analyze some features and geometric aspects of binary decompositions.

The sum of powers decomposition of forms has been widely studied because of its many applications: it is connected with Polynomial Interpolation [4], Fat Points and higher secants of the Veronese varieties [6]. The well-known Waring problem for forms asks the following: determine the minimal number of dth powers needed for the decomposition of the generic form of degree d in $n + 1$ variables. Clearly, as a parameter count shows, the generic form requires a number of summands at least equal to

$$\left\lceil \frac{1}{n+1}\binom{n+d}{d} \right\rceil$$

and, quite surprisingly, this number of summands always suffice with few exceptions:

Theorem 1 (*Alexander and Hirschowitz [1]*). *A generic form of degree d in $n+1$ variables is the sum of*

$$s = \left\lceil \frac{1}{n+1}\binom{n+d}{d} \right\rceil$$

powers of linear forms, unless

- $d = 2$, *where* $s = n + 1$ *instead of* $\lceil \frac{n+2}{2} \rceil$, *or*
- $d = 4$ *and* $n = 2, 3, 4$, *where* $s = 6, 10, 15$ *instead of* $5, 9, 14$ *respectively*, *or*
- $d = 3$ *and* $n = 4$, *where* $s = 8$ *instead of* 7.

The connection between binary decompositions and geometric integration theory was recently brought to my attention by Robert McLachlan [9]. A Waring type question can be asked also in this case: determine the minimal number $s_{\min}(n, d)$ of binary forms appearing in the decomposition of the generic form of degree d in $n + 1$ variables. A parameter count readily gives

$$s_{\min}(n, d) \geqslant \left\lceil \frac{1}{d+2n-1}\binom{n+d}{d} \right\rceil = s_{\exp}(n, d).$$

Whenever n is clear from the context, the shorthand notation $s_{\min}(d)$ and $s_{\exp}(d)$ will be used. Notice that, for forms in three variables, $s_{\exp}(d) = \lceil \frac{d+1}{2} \rceil$. There is no analogue of Theorem 1 for binary decompositions, and only partial results are known for forms in more than three variables. In Carlini [2], it is shown that $s_{\min}(n, d) = s_{\exp}(n, d)$ for the pairs $(3, d)$, $d \leqslant 5$, and $(n, 3)$, any n. In this paper, we give a complete answer to the Waring problem for $n = 2$ by computing $s_{\min}(d)$.

Theorem 2. *The generic form of degree d in three variables is the sum of*

$$\min\left\{s : 2s - \binom{d-s+2}{2} \geqslant 0\right\}$$

binary forms and no fewer.

Given a form $f \in S_d$ and an integer s, and suppose that we have a sum of powers decomposition $f = l_1^d + \cdots + l_s^d$. It is natural to consider the following question: how can one describe all the possible sum of powers decompositions of f involving s summands? Apolarity gives a suitable framework for this investigation: let $T = \mathbb{C}[\partial_0, \ldots, \partial_n]$ and give S a T-module structure via differentiation. As S_1 and T_1 are dual with respect to this action, we set $\mathbb{P}^n = \mathbb{P}T_1$ and $\check{\mathbb{P}}^n = \mathbb{P}S_1$. Denoting by $F = V(f) \subset \mathbb{P}^n$ the hypersurface defined by f, we introduce the *v*ariety of *s*ums of *p*owers (VSP):

$$VSP(F, s) = \overline{\{\mathbb{X} : \mathbb{X} = \{l_1, \ldots, l_s\} \subset \check{\mathbb{P}}^n,\ f = l_1^d + \cdots + l_s^d\}},$$

which is naturally a subscheme of the Hilbert scheme of the sets of s points in $\check{\mathbb{P}}^n$, $\mathrm{Hilb}_s(\check{\mathbb{P}}^n)$. The schemes $VSP(F, s)$ attracted an enormous amount of work in the last decades of 19th century [14,16,17]. Recently, interest in their study has been renewed by works of Mukai [10,11]. Motivated by Mukai's papers, Ranestad and Schreyer [13] attempted the first systematic study of these schemes, but much is still unknown, e.g., on their global structure or their degree. The state of art for forms in three variables is described in Theorem 13.

In this paper, we consider the corresponding problem for binary decompositions. Given a form f let $f = f_1 + \cdots + f_s$ be a binary decomposition and let $l_i \subset \check{\mathbb{P}}^n$, $i = 1, \ldots, s$, be the line spanned by the variables of $f_i(\cdot, \cdot)$. Setting $F = V(f)$, it is natural to define the *v*ariety of *s*ums of *b*inaries (VSB):

$$VSB(F, s) = \overline{\{\Gamma : \Gamma = l_1 \cup \cdots \cup l_s \subset \check{\mathbb{P}}^n,\ f = f_1(\cdot, \cdot) + \cdots + f_s(\cdot, \cdot)\}},$$

which is a subscheme of $\mathrm{Hilb}_s(G(1, \check{\mathbb{P}}^n))$, where $G(1, \check{\mathbb{P}}^n)$ is the Grassmannian of lines in $\check{\mathbb{P}}^n$. For forms in three variables, i.e. $n = 2$, $\mathrm{Hilb}_s(G(1, \check{\mathbb{P}}^n))$ can be described as the *variety of totally decomposable forms*

$$X_s = \overline{\{D : D = L_1 \cdot \ldots \cdot L_s,\ L_i \text{ is a linear form } i = 1, \ldots, s\}} \subset \mathbb{P}T_s$$

and we study the schemes *VSB* in this setting. The main result of this paper is:

Theorem 3. *Let F be a generic plane curve of degree d and let $s = s_{\min}(d)$. Then the following hold for any d*:

(1) $\dim VSB(F, s) = 2s - \binom{d-s+2}{2}$;
(2) $\deg VSB(F, s) = (2s - 1) \cdot (2s - 3) \cdot \cdots \cdot 3$.

For plane curves of degree $d = 2, 3, 5, 8$ a more detailed description of the schemes *VSB* is possible, see Corollary 12.

Notation. We use the identifications $\mathbb{P}^n = \mathbb{P}T_1$ and $\check{\mathbb{P}}^n = \mathbb{P}S_1$. A form is usually denoted with a lower case letter $f \in S_d$ and the hypersurface it defines is denoted with the corresponding capital letter $F = V(f) \subset \mathbb{P}^n$. The opposite for a derivation $G \in T_d$ and the corresponding hypersurface $g = V(G) \subset \check{\mathbb{P}}^n$. The action of G on f is denoted by $G \circ f$ and the orthogonal ideal of f is $f^\perp = \{G : G \circ f = 0\} \subset T$. If $F = V(f)$, then $F^\perp \subset \mathbb{P}T_s$ denotes the projectivization of the vector space $(f^\perp)_s$.

2. Binary decompositions

The main tool for the study of sum of powers decompositions is the classic Apolarity Lemma (see e.g. [13]):

Lemma 4 (*Apolarity Lemma*). *Let $f \in S_d$, then the following are equivalent*:

(1) $f = l_1^d + \cdots + l_s^d$, *where the l_i's are non-proportional linear forms*;
(2) $f^\perp \supset I_{\mathbb{X}}$, *where $I_{\mathbb{X}}$ is the ideal of the set of s points $\mathbb{X} = \{l_1, \ldots, l_s\} \subset \check{\mathbb{P}}^n$.*

If $f = f_1 + \cdots + f_s$ is a binary decomposition, let $l_i \subset \check{\mathbb{P}}^n$ be the line spanned by the variables of $f_i(\cdot,\cdot)$, $i = 1, \ldots, s$, and Γ be the union of these lines. Then it is clear that any $D \in I_\Gamma$ is such that $D \circ f = 0$ and hence $I_\Gamma \subset f^\perp$. Actually, even the converse is true:

Lemma 5 (*Binary Apolarity Lemma*). *Let $f \in S_d$, then the following are equivalent*:

(1) $f = f_1 + \cdots + f_s$, *is a binary decomposition such that the span of the variables of f_i and f_j are distinct in S_1 for $i \neq j$*;
(2) $f^\perp \supset I_\Gamma$, *where I_Γ is the ideal of a set of s distinct lines in $\check{\mathbb{P}}^n$.*

Proof. Let $\Gamma = l_1 \cup \cdots \cup l_s$ and choose $d+1$ distinct points on each line, $p_{0i}, \ldots, p_{di} \in l_i$ for $i = 1, \ldots, s$. Consider the set of $s \cdot (d+1)$ points $\mathbb{X} = \{p_{ij}\}_{ij}$ and notice that $(I_{\mathbb{X}})_i = (I_\Gamma)_i$ for $i \leqslant d$. As $T/f^\perp$ is artinian with socle degree d, the inclusion $I_{\mathbb{X}} \subset f^\perp$ holds and hence

$$f = \sum_{ij} (p_{ij})^d = \sum_{i=1}^{s} \left(\sum_{j=0}^{d} (p_{ji})^d \right)$$

and the latter is a binary decomposition of f. $\square$

Using the Binary Apolarity Lemma, one can easily determine $s_{\min}(2, d)$, i.e. the minimal number of binary forms appearing in the decomposition of the generic form of degree d in three variables.

Theorem 6. *The generic form of degree d in three variables is the sum of* $\min\{s : 2s - \binom{d-s+2}{2} \geqslant 0\}$ *binary forms and no fewer.*

Proof. By Lemma 5 a form f in three variables is the sum of s binary forms if and only if $f^{\perp}$ contains a totally decomposable form of degree s without repeated factors. Consider the incidence correspondence

$$\Sigma = \{(f, D) : D \in F^{\perp}, D = L_1 \cdot \ldots \cdot L_s\} \subset \mathbb{P}S_d \times X_s$$

and the incidence diagram

$$\begin{tikzcd} & \Sigma \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}S_d & & X_S. \end{tikzcd}$$

Clearly $\dim \Sigma = \dim \mathbb{P}S_d + 2s - \binom{d-s+2}{2}$ (use β to show that Σ is a projective bundle over X_s). Moreover, for $s \geqslant \bar{s} = \min\{t : 2t - \binom{d-t+2}{2} \geqslant 0\}$, the map α is surjective (a dimension count shows that $X_s \cap F^{\perp} \neq \emptyset$ for any f). Let $\Sigma_0 = \{(f, D = L_1 \cdot \ldots \cdot L_s) : D \in F^{\perp}, L_i \sim L_j$ for some $i \neq j\}$ and notice that Σ_0 has codimension 2 in Σ. For $s \geqslant \bar{s}$ and f generic, a dimension argument yields $\alpha^{-1}(f) \backslash \Sigma_0 \neq \emptyset$, hence the claim. □

As a consequence, we get a comparison between the expected number of summands and the number actually needed, for a proof see [2, Chapter 3]. Notice the sharp contrast with the sum of powers case where only few exceptions exist.

Corollary 7. *The generic form of degree d in three variables is the sum of the expected number of binary forms if and only if* $d = 2, 3, 4, 5, 6$ *and* 8.

When more than three variables are involved, i.e. $n \geqslant 3$, only partial results are known on $s_{\min}$ (see [2]): namely, $s_{\min}(n, 3) = s_{\exp}(n, 3)$ for any n and $s_{\min}(3, d) = s_{\exp}(3, d) = 2, 3, 4, 6$ for $d = 2, 3, 4, 5$. The results for $n = 3$ and $d = 2, 3$ can be easily explained: the quadratic case is just a diagonalization argument; for the cubic case, recall that the generic form of degree 3 in four variables is the sum of 5 powers of linear forms, $l_1^3 + \cdots + l_5^3$, and hence the sum of 3 binary forms $l_1^3 + l_2^3$, $l_3^3 + l_4^3$ and l_5^3.

3. VSB for forms in three variables

We have already noted that, for a given plane curve $F = V(f)$, the scheme $VSB(F, s)$ can be described as a subscheme of a projective space as follows:

$$VSB(F, s) = X_s \cap F^{\perp} \subset \mathbb{P}T_s,$$

where X_s is the variety of totally decomposable forms of degree s in three variables.

Remark 8. Notice that X_s is singular in codimension 2. Hence $VSB \subset \mathbb{P}T_s$ is singular as soon as $\dim VSB \geqslant 2$. This is clearly a drawback of this description, nevertheless, for $d \leqslant 8$ and $\dim VSB = 0, 1$, one is able to show that the schemes are smooth.

As the variety X_s plays a central role, we devote some time investigating some of its properties. In particular, we describe $T_P(X_s)$, i.e. the tangent space to X_s in a generic point P, and we determine $\deg X_s$ in an elementary way. In Chipalkatti [3], an invariant theory approach is used and the homogeneous saturated ideal of X_s is described for $s = 3$, but no explicit formula for the degree is given.

Recall that $l \subset \check{\mathbb{P}}^2$ denotes the line defined by $L \in T_1$.

Lemma 9. *Let $P = L_1 \cdot \ldots \cdot L_s \in X_s$ be a generic point and set $\Gamma_P = \{Q \in \check{\mathbb{P}}^2 : Q \in l_i \cap l_j$ for some $i \neq j\}$. Then*

$$T_P(X_s) = \left| sH - \sum_{Q \in \Gamma_P} Q \right| = \{curves\ of\ degree\ s\ containing\ \Gamma_P\}.$$

Proof. As P is a generic point $\dim_{\mathbb{C}} T_P(X_s) = 2s+1$. If we set $\mathscr{L}_P = |sH - \sum_{Q \in \Gamma_P} Q|$, then $T_P(X_s) \subseteq \mathscr{L}_P$ (to see this use a parameterization of X_s). Hence we only have to determine $\dim_{\mathbb{C}} \mathscr{L}_P = \binom{s+2}{2} - H_{\Gamma_P}(s)$, where H_{Γ_P} denotes the Hilbert function. To determine the Hilbert function of the set of $\binom{s}{2}$ points Γ_P, we notice that there is no curve of degree less than $s-1$ containing the points and hence $H_{\Gamma_P}(s-1) = \binom{s}{2} = H_{\Gamma_P}(s)$ (recall that the Hilbert function of a set of points is not decreasing, [7]). We conclude that $\dim_{\mathbb{C}} \mathscr{L}_P = \dim_{\mathbb{C}} T_P(X_s)$ and hence the claim. $\square$

As $\dim X_s = 2s$, to determine $\deg X_s$ we have to intersect the variety with a linear space H of codimension $2s$. There is a natural way to choose a codimension t linear space in $\mathbb{P}T_s$: choose generic points $P_1, \ldots, P_t \in \check{\mathbb{P}}^2$ and consider the curves of degree s passing through them, they form a linear space $H(P_1, \ldots, P_t) \subset \mathbb{P}T_s$ of codimension t. We exploit this simple idea in the following:

Proposition 10. *The variety of totally decomposable forms of degree s in three variables has degree $(2s-1) \cdot (2s-3) \cdot \cdots \cdot 3$, i.e. the number of s- uples of lines passing through $2s$ generic points in the plane.*

Proof. Choose a set of generic points $\mathbb{X} = \{P_1, \ldots, P_{2s}\} \subset \check{\mathbb{P}}^2$ and let $H = H(P_1, \ldots, P_{2s}) \subset \mathbb{P}T_s$ be the linear space that they determine. The scheme $Z = H \cap X_s$ is zero-dimensional by the genericity of $\mathbb{X}$ and its support is the set of s- tuples of lines through $\mathbb{X}$. If Z is smooth, then the claim is proved. Notice that $P \in Z$ is a smooth point if and only if $H \cap T_P(X_s) = \{P\}$. To check this, let $P = L_1 \cdot \ldots \cdot L_s$ and consider the set of points $\mathbb{Y} = \{Q_1, \ldots, Q_{\binom{s}{2}}\} = \{Q : Q \in l_i \cap l_j$ for some $i \neq j\} \subset \check{\mathbb{P}}^2$, then Lemma 9 yields:

$$\begin{aligned} H \cap T_P(X_s) &= \left| sH - \sum P_i - \sum Q_i \right| \\ &= \{\text{curves of degree } s \text{ passing through } \mathbb{X} \text{ and } \mathbb{Y}\}. \end{aligned}$$

Notice that, by the genericity of $\mathbb{X}$, $\mathbb{X}\cup\mathbb{Y}$ can be partitioned as follows: there are $(s-1)+2$ points on l_1, there are $(s-2)+2$ points on l_2, etc. Hence $l_1\cup\cdots\cdots\cup l_s$ is the only curve of degree s containing $\mathbb{X}\cup\mathbb{Y}$ and the claim follows. $\square$

Finally, we can prove the main result.

Theorem 11. *Let F be a generic plane curve of degree d and let $s=s_{\min}(d)$. Then the following hold for any d*:

(1) $\dim VSB(F,s)=2s-\binom{d-s+2}{2}$;
(2) $\deg VSB(F,s)=(2s-1)\cdot(2s-3)\cdot\cdots\cdot 3$.

Proof. Use the notation of the proof of Theorem 6 and notice that, for a generic $F=V(f)$, $\alpha^{-1}(f)=VSB(F,s)$. Claim (1) follows from a dimension argument. As F is generic (and applying (1)), the intersection $X_s\cap F^{\perp}$ is seen to be proper and hence $\deg VSB(F,s)=\deg X_s$. $\square$

The description of *VSB* given by the previous theorem is quite coarse, but some refinements are possible in special cases, namely for $d=2,3,5,8$. We give a description of $VSB(F,s_{\min}(d))$ for plane curves of degree $d\leqslant 8$ in Corollary 12 which should be viewed as a first approximation to Theorem 13 for the *VSP*s.

Corollary 12. *Let F be a general plane curve of degree $d\leqslant 8$.*

(1) *If $d=2$, then $VSB(F,2)$ is the variety of secant lines to the rational normal curve in $\mathbb{P}^4$.*
(2) *If $d=3$, then $VSB(F,2)$ is a smooth plane cubic.*
(3) *If $d=4$, then $VSB(F,3)$ is a singular 3-fold in $\mathbb{P}^6$ of degree 15.*
(4) *If $d=5$, then $VSB(F,3)$ is finite and smooth of degree 15 in $\mathbb{P}^3$, i.e. is a set of 15 points.*
(5) *If $d=6$, then $VSB(F,4)$ is a singular surface in $\mathbb{P}^8$ of degree 105.*
(6) *If $d=7$, then $VSB(F,5)$ is a singular 4-fold in $\mathbb{P}^{14}$ of degree 945.*
(7) *If $d=8$, then $VSB(F,5)$ is finite and smooth of degree 945 in $\mathbb{P}^{10}$, i.e. is a set of 945 points.*

Proof. For $d=2$, $VSB(F,2)=X_2\cap F^{\perp}\subset\mathbb{P}^5$ and the variety of totally decomposable forms coincides with the variety of secant lines to the Veronese surface. Hence X_2 is a hypersurface of degree 3 smooth outside the Veronese. Moreover, by the genericity of the curve, $F^{\perp}$ is a generic hyperplane and $VSB(F,2)$ is a generic hyperplane section of X_2 and the claim follows.

For $d=3$, again $VSB(F,2)=X_2\cap F^{\perp}\subset\mathbb{P}^5$ and $F^{\perp}$ is a generic $\mathbb{P}^2\subset\mathbb{P}^5$. Hence the claim.

For $d=5$, $VSB(F,3)=X_3\cap F^{\perp}\subset\mathbb{P}^9$ and $F^{\perp}$ is a $\mathbb{P}^3$ which could very well not be generic as $\dim G(3,\mathbb{P}^9)=24>\dim\mathbb{P}S_5=20$. So we have to check that the zero-dimensional scheme $Z=VSB(F,3)$ of degree 15 is smooth for a generic choice of F. As the non-smoothness condition on Z is a closed condition on F, we only have to produce a particular g such that $VSB(G = V(g), 3)$ consists of exactly 15 points. Choose six linear forms $l_1, \ldots, l_6 \in S_1$ and let I be the ideal of the corresponding points in $\check{\mathbb{P}}^2$. Let $g = l_1^5 + \cdots\cdots + l_6^5$ and notice that $(g^\perp)_3 \supseteq I_3$. Hence $VSB(G, 3)$ contains 15 distinct points, i.e. the 15 triples of lines through the six points in $\check{\mathbb{P}}^2$. *If* $VSB(G, 3)$ is zero-dimensional the claim is proved. Notice that, since g is a special form (a generic quintic *is not* the sum of six powers), the zero-dimensionality is not obvious at all. To show that $\dim VSB(G, 3) = 0$, it is enough to show that $(g^\perp)_3 = I_3$ or, equivalently, that $\dim_{\mathbb{C}}(g^\perp)_2 = 0$. Denote by Q_i the conic in $\check{\mathbb{P}}^2$ passing through $l_1, \ldots, \hat{l_i}, \ldots, l_6$ and notice that $Q_1, \ldots, Q_6$ are a basis of T_2. If$(g^\perp)_2 \neq 0$, then there exist $\lambda_1, \ldots, \lambda_6 \in \mathbb{C}$, not all zero, such that

$$\left(\sum_1^6 \lambda_i Q_i\right) \circ g = \sum_1^6 c_i \lambda_i l_i^3 = 0,$$

where the c_i's are not zero. But this is clearly not the case if the l_i's are chosen freely. Hence the claim follows as $VSB(G, 3)$ is a zero-dimensional, smooth scheme of degree 15.

For $d = 8$, the proof is identical to the $d = 5$ case. □

For the convenience of the reader, we summarize what it is known for $VSP(F, s)$ in the case of plane curves of degree $d \leqslant 8$; for more details see Ranestad and Schreyer [13].

Theorem 13. *Let F be a general plane curve of degree* $d \leqslant 8$.

(1) *Mukai* [10]. *If* $d = 2$, *then* $VSP(F, 3)$ *is a Fano* 3*-fold of index* 2 *and degree* 5 *in* $\mathbb{P}^6$.
(2) *If* $d = 3$, *then* $VSP(F, 4)$ *is the projective plane*.
(3) *Mukai* [10]. *If* $d = 4$, *then* $VSP(F, 6) \simeq V_{22}$, *i.e. a Fano* 3*-fold of index* 1 *and genus* 12 *with anticanonical embedding of degree* 22.
(4) *Hilbert* [8], *Palatini* [12], *and Richmond* [15]. *If* $d = 5$, *then* $VSP(F, 7)$ *is finite and smooth of degree* 1, *i.e. consists of a single point*.
(5) *Mukai* [11]. *If* $d = 6$, *then* $VSP(F, 10)$ *is isomorphic to a polarized* $K3$*-surface of genus* 20.
(6) *Dixon and Stuart* [5]. *If* $d = 7$, *then* $VSP(F, 12)$ *is finite and smooth of degree* 5, *i.e. consists of* 5 *distinct points*.
(7) *Ranestad and Schreyer* [13]. *If* $d = 8$, *then* $VSP(F, 15)$ *is finite and smooth of degree* 16, *i.e. consists of* 16 *distinct points*.

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